The quark and gluon condensates and low-energy QCD theorems in a magnetic field

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Abstract

The low-energy QCD theorems are generalized in the presence of a constant magnetic field H. Two-loop approximation for the vacuum energy density in the framework of the chiral perturbation theory was obtained and the quark and gluon condensates were found as the functions of H.

1 Introduction.

The low-energy theorems, playing an important role in the understanding the vacuum state properties in quantum field theory, were discovered almost at the same as quantum field methods appeared in particle physics (see, for example, Low theorems [1]). In QCD, these theorems were discovered in the beginning of eighties [2]. These theorems, being derived from the very general symmetrical considerations and not depending on the details of confinement mechanism, sometimes gives information which is not easy to obtain in another way. Also, they can be used as "physically sensible" restrictions in the constructing of effective theories. Recently, they were generalized to finite temperature and chemical potential case [3].

The behavior of the quark condensate in the presence of a magnetic field was studied in Nambu-Jona-Lasinio model earlier [4]. For QCD, the analogous investigation in one-loop approximations was done in [5]. It was found that the quark condensate grows with the increase of the magnetic field H in both cases. It implies that naive analogy with superconductivity, where the order parameter vanishes at some critical field, is not valid here.

In this paper, we generalize the low-energy theorems to the case of the presence of the constant magnetic field. When the field is weak (compared to the characteristic

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hadronic scale), two-loop formulae for the vacuum energy density, the quark and gluon condensates as the functions of H were obtained basing on the chiral perturbation theory (ChPT). Gluons do not carry electric charge, nevertheless, virtual quarks produced by them interact with electromagnetic field and lead to the changes in the gluon condensate.

2 Low-energy theorems in the magnetic field.

The Euclidean version of QCD partition function in the magnetic field A_μ can be written as follows

$$Z = \exp\left\{-\frac{1}{4}\int d^4xF^2_{\mu\nu}\right\}\int[\mathcal{D}B^a_\mu][\mathcal{D}\bar{q}_f][\mathcal{D}q_f]\exp\{-\int d^4xL_{QCD}\}, \tag{1}$$

where QCD Lagrangian in the background field is

$$L_{QCD} = \frac{1}{4g_0^2}(G^a_{\mu\nu})^2 + \sum_f \bar{q}_f\left[\gamma_\mu(\partial_\mu - iQ_feA_\mu - i\frac{\lambda^a}{2}B^a_\mu) + m_f\right]q_f, \tag{2}$$

Here Q_f is the charge matrix of quarks in the flavor space and we have suppressed gauge fixed and Faddeev-Popov terms since they are unessential here. The vacuum energy density is defined in the usual way, $V_4\epsilon_v(H, m_f) = -\ln Z$. In the constant magnetic field, the chiral condensate in the chiral limit ($m_f \to 0$) is given by

$$\langle\bar{q}_fq_f\rangle(H) = \left.\frac{\partial\epsilon_v(H, m_f)}{\partial m_f}\right|_{m_f=0} \tag{3}$$

From (1) follows that the gluon condensate $\langle G^2\rangle \equiv \langle(G^a_{\mu\nu})^2\rangle$ can be written in the following form

$$\langle G^2\rangle = 4\left.\frac{\partial\epsilon_v(H, m_f)}{\partial(1/g_0^2)}\right|_{m_f=0} \tag{4}$$

The effect of dimensional transmutation produces a new dimensionful nonperturbative parameter

$$\Lambda = M\exp\left\{\int_{\alpha_s(M)}^{\infty}\frac{d\alpha_s}{\beta(\alpha_s)}\right\}, \tag{5}$$

where M - is ultraviolet cut-off, $\alpha_s = g_0^2/4\pi$, and $\beta(\alpha_s) = d\alpha_s(M)/d\ln M$ is Gell-Mann - Low function.

The system, corresponding to partition function (1), contains two dimensionful parameters M and H in the chiral limit ($m_f = 0$). As the vacuum energy density is an observable quantity, it is also renorm-invariant one and its anomalous dimension is zero. Therefore, ϵ_v has only normal (canonical) dimension which equals four. Using the renorm-invariance of Λ, we can write the most general expression for ϵ_v

$$\epsilon_v = \Lambda^4 f(H/\Lambda^2), \tag{6}$$

where f is some function. Then, it is easy to get from Eqs. (5) and (6) that

$$\frac{\partial\epsilon_v}{\partial(1/g_0^2)} = \frac{8\pi\alpha_s^2}{\beta(\alpha_s)}\left(2 - H\frac{\partial}{\partial H}\right)\epsilon_v, \tag{7}$$

and from Eq. (4), we can find the relation between ϵ_v and the gluon condensate

$$\langle G^2\rangle(H) = \frac{32\pi\alpha_s^2}{\beta(\alpha_s)}\left(2 - H\frac{\partial}{\partial H}\right)\epsilon_v \tag{8}$$

If we set $H = 0$ we would reproduce the well-known expression for nonperturbative density of the vacuum energy in the chiral limit. In one-loop approximation ($\beta = -b_0\alpha_s^2/2\pi$, $b_0 = (11N_c - 2N_f)/3$), it reads

$$\epsilon_v = -\frac{b_0}{128\pi^2}\langle G^2\rangle. \tag{9}$$

Let us now derive the low-energy theorems in the presence of magnetic field. We can iterate the procedure described above by taking n further derivatives of Eq. (4)

$$(-1)^n\left(4 - 2H\frac{\partial}{\partial H}\right)^{n+1}\epsilon_v = \left(2H\frac{\partial}{\partial H} - 4\right)^n\langle\sigma\rangle =$$
$$= \int d^4x_n\cdots d^4x_1\langle\sigma(x_n)\cdots\sigma(x_1)\rangle_c. \tag{10}$$

Here

$$\sigma(x) = \theta_{\mu\mu}(x) = \frac{\beta(\alpha_s)}{16\pi\alpha_s^2}(G^a_{\mu\nu}(x))^2, \tag{11}$$

and the subscript 'c' means that only connected diagrams are to be included. Proceeding in the same way, it is possible to obtain the similar theorems for renorm-invariant operator $O(x)$ which is built from quark and/or gluon fields

$$\left(2H\frac{\partial}{\partial H} - d\right)^n\langle O\rangle = \int d^4x_n\cdots d^4x_1\langle\sigma(x_n)\cdots\sigma(x_1)O(0)\rangle_c, \tag{12}$$

where d - is canonical dimension of operator O. If operator O has also anomalous dimension, the appropriate γ-function should be included.

3 Vacuum energy in the magnetic field.

The obtained results permit us to calculate the condensates in the chiral limit as functions of H, once the vacuum energy density is known. To calculate ϵ_v, we have to consider the loops in the magnetic field. When the field is weak, $eH << \mu^2_{hadr} \sim (4\pi F_\pi)^2$, characteristic momenta in loops are small and theory is adequately described by the low-energy effective Lagrangian [6] which admits an expansion in powers of the external momenta (derivatives) and the quark masses [3]

$$L_{eff} = L^{(2)} + L^{(4)} + L^{(6)} + \cdots \tag{13}$$

The leading term $L^{(2)}$ in (13) is similar to the Lagrangian of nonlinear σ-model in external field V_μ

$$L^{(2)} = \frac{F_\pi^2}{4}\mathrm{Tr}(\nabla_\mu U^+\nabla_\mu U) + \Sigma\mathrm{ReTr}(\mathcal{M}U^+), \tag{14}$$

[3] Note that the chiral limit means $eH >> M_\pi^2$.

$$\nabla_\mu U = \partial_\mu U - i\,[U, V_\mu]\,.$$

Here U stands for an unitary $SU(2)$ matrix, $F_\pi = 93$ Mev is the pion decay constant and parameter Σ has the meaning of the quark condensate $\Sigma = |\langle \bar{u}u\rangle| = |\langle \bar{d}d\rangle|$. The external Abelian magnetic field H, aligned along z-axis corresponds to $V_\mu(x) = e(\tau^3/2)A_\mu(x)$, where vector potential A_μ may be chosen as $A_1(x) = -Hx_2$.

The difference of light quark mass $m_u - m_d$ enters in the effective Lagrangian (13) only quadratically. To calculate the fermion condensate, we need to take only first derivative over the quark mass before going to the chiral limit. In the case of the gluon condensate, we can put $m_u = m_d = 0$ from the very beginning. It means that we can take mass matrix to be diagonal $\mathcal{M} = m\hat{I}$.

In the leading approximation the Lagrangian (14) coincides with the Lagrangian of scalar electrodynamics. In one-loop approximation, the result for the vacuum energy density in this theory was found long time ago by Schwinger [7]

$$\epsilon_v^{(1)}(H) = -\frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-M_\pi^2 s}\left[\frac{eHs}{\sinh eHs} - 1\right] \tag{15}$$

Using Eqs. (3), (8) and Gell-Mann - Oakes - Renner relation (GMOR) $2m\Sigma = F_\pi^2 M_\pi^2$, we can get one-loop formulae for the condensates in the constant magnetic field

$$\Sigma(H) = \Sigma\left[1 + \frac{eH\ln 2}{(4\pi F_\pi)^2}\right], \tag{16}$$

$$\langle G^2\rangle(H) = \langle G^2\rangle + \frac{\alpha_s^2}{3\pi\beta(\alpha_s)}(eH)^2. \tag{17}$$

In the chiral perturbation theory in the magnetic field, the parameter of the expansion is $eH/(4\pi F_\pi)^2$. To find ϵ_v to the next order or the expansion, we have to take into account two-loop diagrams with the vertices from $L^{(2)}$, one-loop diagrams with one vertex from $L^{(4)}$ and a tree contribution arising from $L^{(6)}$. The corresponded Feynman graphs for $\epsilon_v^{(2)}$ are depicted in Fig. 1.

To perform the calculations of these diagrams, we should expand $L^{(2)}$ up to four-pion vertices. The matrix U can be parameterized in many ways. We choose Weinberg parameterization

$$U = \sigma + \frac{i\pi^a\tau^a}{F_\pi}, \quad \sigma^2 + \frac{\vec{\pi}^2}{F_\pi^2} = 1. \tag{18}$$

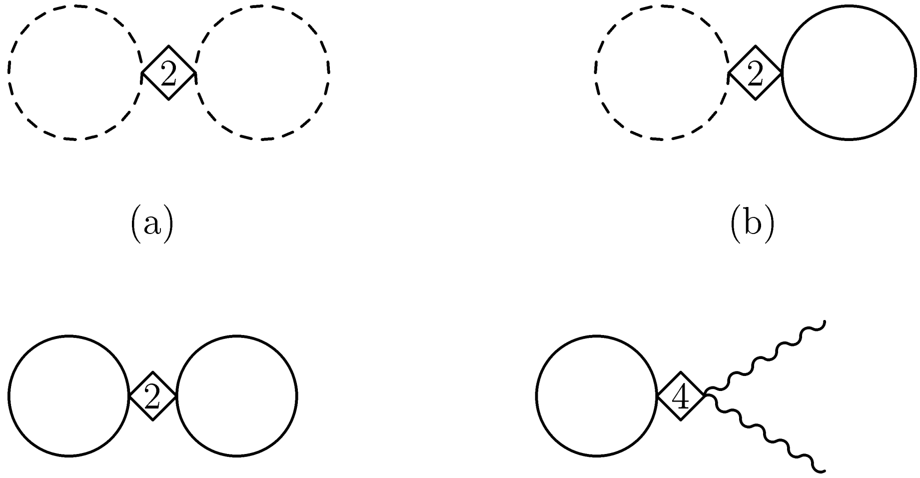
Then the expansion of $L^{(2)}$ is

$$L^{(2)} = \frac{1}{2}(\partial_\mu\pi^0)^2 - \frac{M_\pi^2(\pi^0)^2}{2} + (\partial_\mu\pi^+ + ieA_\mu\pi^+)(\partial_\mu\pi^- - ieA_\mu\pi^-) - M_\pi^2\pi^+\pi^-$$

$$+\frac{1}{2F_\pi^2}[\pi^0\partial_\mu\pi^0 + \partial_\mu(\pi^+\pi^-)]^2 - \frac{M_\pi^2}{8F_\pi^2}[2\pi^+\pi^- + (\pi^0)^2]^2, \tag{19}$$

where we have introduced the fields of the charged and the neutral pions

$$\pi^0 = \pi^3, \quad \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2). \tag{20}$$



(a) (b) (c) (d)

Figure 1: The loop diagrams contribute to vacuum energy density to the second order of ChPT.

We need also the expression of a charged pion propagator in a magnetic field. It can be inferred from the results of Ref. [8] where an explicit expression for the fermion propagator in a magnetic field at nonzero chemical potential μ has been found. After some simplifications, we arrive at the following formula for the Euclidean scalar propagator

$$D^H(x,y) = \Phi(x,y) \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} D^H(k), \tag{21}$$

where $\Phi(x,y) = \exp\{ie \int_y^x A_\mu(z) dz_\mu\}$ is the phase factor and the integration is done along the straight line connecting x and y, and

$$D^H(k) = \int_0^\infty \frac{ds}{\cosh(eHs)} \exp\left\{ -s \left(k_\parallel^2 + k_\perp^2 \frac{\tanh eHs}{eHs} + M_\pi^2 \right) \right\} \tag{22}$$

with $k_\parallel^2 = k_3^2 + k_4^2$, $k_\perp^2 = k_1^2 + k_2^2$.

The contribution of the graph in Fig. 1a to ϵ_v is

$$\epsilon_v^{(2)}[\text{Fig.1a}] = -\frac{M_\pi^2}{8F_\pi^2} D^2(0), \tag{23}$$

where $D(0)$ is a free propagator of the scalar particle at coinciding points which can be written in dimensional regularization scheme as

$$D(0) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + M^2} = 2M^2 \left(\lambda + \frac{1}{32\pi^2} \ln \frac{M^2}{\mu^2} \right), \tag{24}$$

where μ is the mass parameter of the regularization and a singular term in (24) is explicitely separated out as

$$\lambda = \frac{\mu^{d-4}}{16\pi^2} \left[\frac{1}{d-4} - \frac{1}{2} (\ln 4\pi - \gamma_E + 1) \right]. \tag{25}$$

The correction (23) can be absorbed to ϵ_v at $H = 0$ and does not cause the shifts of the condensates by magnetic field.

The correction to ϵ_v, coming from the next diagram Fig. 1b, is

$$\epsilon_v^{(2)}[\text{Fig.1b}] = \frac{M_\pi^2}{2F_\pi^2} D(0) D^H(0), \tag{26}$$

where $D^H(0) = D^H(x,x)$. According to GMOR, the result in (26) is proportional to the square of the quark mass and does not change the condensates in the chiral limit.

Performing the calculation of the diagram Fig. 1c, we get

$$\epsilon_v^{(2)}[\text{Fig.1c}] = \frac{1}{F_\pi^2} D^H(0) \int \frac{d^d k}{(2\pi)^d} (k^2 + M_\pi^2) D^H(k). \tag{27}$$

This expression contains a quartic divergence. However, by virtue of the dimensional regularization identity $\int d^d k = 0$, this divergence can be ignored [4]. Subtracting one from the integrand in (27) and going to a limit $d \to 4$, we find that $\epsilon_v^{(2)}[Fig.1c] = 0$. So, we arrive at the conclusion that two-loop diagrams do not give the correction to ϵ_v, linear in the quark mass and, therefore, do not shift the condensates.

In this order of expansion in $eH/(4\pi F_\pi)^2$, there are some diagrams with vertices from $L^{(4)}$ besides the diagrams discussed above. As the momentum of a constant field is zero, only the two terms from the general form of $L^{(4)}$ are left

$$L^{(4)} = -\frac{2l_5}{F_\pi^2}(eF_{\mu\nu})^2 \pi^+\pi^- - \frac{2il_6}{F_\pi^2} eF_{\mu\nu}[\partial_\mu \pi^- \partial_\nu \pi^+ + ieA_\mu \partial_\nu(\pi^+\pi^-)], \tag{28}$$

where the phenomenological constants (infinite) l_5 l_6 were defined in [6]. The corresponding diagrams are drawn in Fig. 1d. The calculation is straightforward and gives

$$\epsilon_v^{(2)}[\text{Fig.1d}] = \frac{2(eH)^2}{F_\pi^2}(2l_5 - l_6) D^H(0). \tag{29}$$

Although the constants l_5 l_6 are infinite, their combination, arising in (29) is finite [6]

$$2l_5 - l_6 = \frac{1}{96\pi^2}(\bar{l}_6 - \bar{l}_5), \tag{30}$$

and $\bar{l}_6 - \bar{l}_5 \approx 2.7$.

Total expression $L^{(6)}$, given by chiral symmetry, is rather complicated [10]. However, for our purposes, the only one term is important which can be taken in the following form, convenient for us

$$L^{(6)} = \frac{80d}{9F_\pi^4}(eF_{\mu\nu})^2 \Sigma \mathrm{ReTr}\{\mathcal{M}U^+\} \tag{31}$$

Here $d = d^r(\mu) + const \cdot \lambda$ and λ involves a pole $\mu^{(d-4)}/(d-4)$. The divergent part in (31) is canceled by the pole terms, stemming from one-loop diagrams with vertices from $L^{(4)}$.

[4] For other regularizations, quartic divergence is removed only if a nontrivial measure for the group integration is taken into account. The examples of explicit calculations can be found in [9].

Numerical value of d^r can be inferred from the results obtained in [11] where the process $\gamma\gamma \to \pi^0\pi^0$ has been studied. In the notations of [11], d^r can be rewritten as

$$d^r = \frac{9}{320}(\bar{a}_1 + 2\bar{a}_2 + 4\bar{b}), \tag{32}$$

where $\bar{a}_1$, $\bar{a}_2$ and $\bar{b}$ have the meaning of the coefficients at different tensor and mass structures in the amplitude of the process $\gamma\gamma \to \pi^0\pi^0$. The numerical values of $\bar{a}_1$, $\bar{a}_2$ and $\bar{b}$ have been obtained by vector, tensor and scalar resonances saturation of the amplitude. [11]. It turns out that only the exchange by scalar mesons gives the contribution to d^r and

$$d^r(\mu \sim 0.75 \quad \mathrm{Gev}) \approx \pm 4 \cdot 10^{-6} \tag{33}$$

The scalar meson-photons and meson-pions coupling constants enter in the experimentally observable decay widths quadratically, since in the effective chiral Lagrangian they emerge linearly. Hence, it easy to see that the sign of d^r remains undetermined.

Now we are in the position to get a finite result for ϵ_v. To find the quark condensate, one should keep only first order terms in M_π^2/eH expansion

$$D^H(0) = [D^H(0) - D(0)] + D(0) \approx -\frac{eH\ln 2}{16\pi^2}$$
$$+\frac{M_\pi^2}{16\pi^2}\left[\ln\frac{eH}{M_\pi^2} + C\right] + 2M_\pi^2\left[\lambda + \frac{1}{32\pi^2}\ln\frac{M_\pi^2}{\mu^2}\right], \tag{34}$$

where C - is slowly varying function of eH/M_π^2 and $C(0) \approx -0.2$. Adding up the obtained results, we arrive at the final answer for ϵ_v in two-loop approximation

$$\epsilon_v(H) = \epsilon_v(0) + \epsilon_v^{(1)}(H) + \frac{1}{48\pi^2}\frac{(eH)^2}{(4\pi F_\pi)^2}(\bar{l}_6 - \bar{l}_5)\left\{-eH\ln 2 + M_\pi^2\left[\ln\frac{eH}{\mu^2} + C\right]\right\}$$
$$-\frac{160 d^r(\mu)}{9F_\pi^2}(eH)^2 M_\pi^2. \tag{35}$$

Using (3), (8) and GMOR relation, it is easy to find how the quark condensate depends upon H

$$\Sigma(H) = \Sigma\left\{1 + \frac{eH}{(4\pi F_\pi)^2}\ln 2 - \frac{1}{3}\frac{(eH)^2}{(4\pi F_\pi)^4}\left[(\bar{l}_6 - \bar{l}_5)\left(\ln\frac{eH}{\mu^2} + C\right)\right.\right.$$
$$\left.\left.-\frac{160(4\pi)^4}{3}d^r(\mu)\right]\right\}. \tag{36}$$

Further, we have for the gluon condensate

$$\langle G^2\rangle(H) = \langle G^2\rangle + \frac{\alpha_s^2}{3\pi\beta(\alpha_s)}(eH)^2\left[1 + 2\frac{eH}{(4\pi F_\pi)^2}(\bar{l}_6 - \bar{l}_5)\ln 2\right]. \tag{37}$$

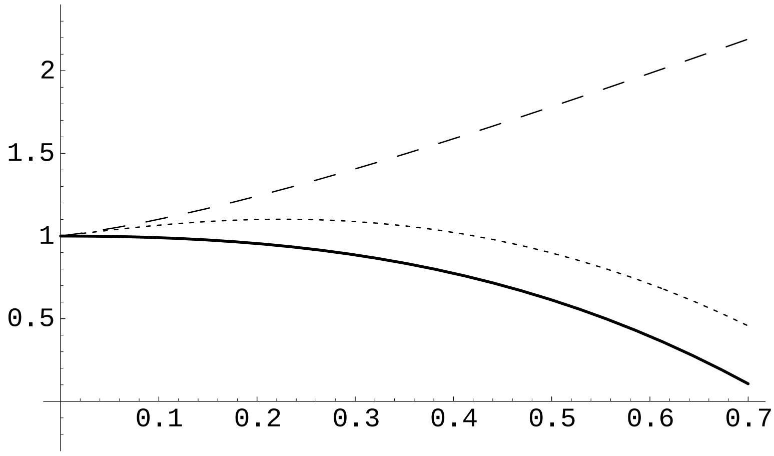


Figure 2: The dependence of the gluon condensate $\langle G^2(x)\rangle/\langle G^2\rangle$ (solid line) and the quark condensate $\Sigma(x)/\Sigma$ (dotted line corresponds to $d^r > 0$ and dashed line corresponds to $d^r < 0$) upon $eH/(4\pi F_\pi)^2$.

It follows from asymptotic freedom that $\beta(\alpha_s) < 0$ and, consequently, the gluon condensate drops when the magnetic field arises. In one loop approximation for QCD β- function ($\beta(\alpha_s) = -b_0\alpha_s^2/2\pi$), this decrease is equal to

$$\Delta\langle G^2\rangle = -\frac{2\pi}{3b_0}(eH)^2\left[1 + 2\frac{eH}{(4\pi F_\pi)^2}(\bar{l}_6 - \bar{l}_5)\ln 2\right] \tag{38}$$

Let us introduce the dimensionless variable $x = eH/(4\pi F_\pi)^2$. Substituting the found numerical values into Eq. (36), we can write the quark condensate as a function of x

$$\Sigma(x)/\Sigma = 1 + x\ln 2 - ax^2\ln x - bx^2. \tag{39}$$

Here $a \simeq 0.9$ and $b \simeq 0.65 \pm 1.77$, where, for the coefficient b, we have used two values of d^r from (33). The behavior of condensates $\langle G^2(x)\rangle/\langle G^2\rangle$ and $\Sigma(x)/\Sigma$ (for two different values of b) is shown in Fig. 2. If $d^r > 0$ then the quark condensate starts to decrease at the magnitude of the magnetic field $x > 0.23$.

4 Conclusions

We have generalized the low-energy QCD theorems in the presence of the constant magnetic field H. Basing on ChPT, we have calculated the vacuum energy density in two-loop approximation and found the dependence of the quark and gluon condensates upon the magnitude of H. It is shown that the gluon condensate grows with the increasing of H while the behavior of the quark condensate crucially depends on the sign of d^r. It increases if we choose positive sign and decreases in the opposite case. Note that possible decreasing of the condensate Σ happens in the region of the applicability of ChPT $eH/(4\pi F_\pi)^2 < 1$. As it was already mentioned, it is not possible to determine the sign of d^r from experimental data. We will not bring here various speculations concerning the behavior of Σ in the magnetic field. We would like only to say that the question about the sign of Σ changing in two-loop level would be resolved if the phenomenological constants of ChPT are derived from the first principles of QCD.

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